Universality of the large deviation principle in one-dimensional dynamics

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2016.3.9. Fractal Geometry, Hyperbolic Dynamics and Thermodynamical Formalism
（ICERM, Brown University）

1. Introduction

Let $X \subset \mathbb{R}$ be a compact interval,

m the (normalized) Lebesgue measure on X as a reference measure,

$f : X \to X$ a smooth map (not necessary to be invariant).

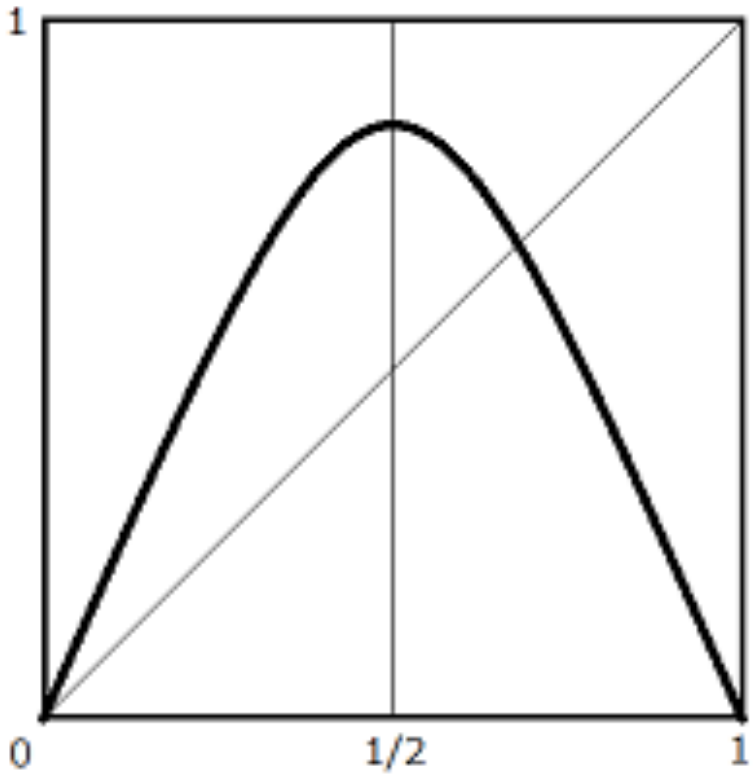
The purpose of study in dynamical systems is to investigate

$$x \in X, \quad f^n(x) := f \circ \cdots \circ f(x) \to ? \quad (n \to \infty)$$

The example in mind is the family of quadratic maps

$$X = [0,1], \; f(x) = f_a(x) = ax(1\text{-}x), \; 0 < a \leq 4.$$

2. The graph of a quadratic map



3. Ergodic Theory

The **empirical distribution**

$$\delta_x^n := \frac{1}{n}(\delta_x + \delta_{f(x)} + \cdots + \delta_{f^{n-1}(x)}) \to ? \quad (n \to \infty)$$

(in other words, the **time average**

$$\frac{1}{n} S_n \varphi(x) : \quad = \quad \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \to ? \quad (n \to \infty)$$

for an observable $\varphi : X \to \mathbf{R}$)

4. Physical measure

$\mu_0 \in \mathcal{M}_f$ is a **physical measure** for f if the set of $x \in X$ with $\delta_x^n \xrightarrow{w^*} \mu_0 \ (n \to \infty)$,

i.e. $\frac{1}{n} S_n \varphi(x) \to \int \varphi \, d\mu_0 \ (n \to \infty), \ \forall \varphi \in C(X),$

has positive Lebesgue measure,

where $\mathcal{M}_f$ denotes the set of f -invariant Borel probability measures on X.

The existence of physical measures corresponds to **LLN** in probability theory.

5. Typical types of physical measures

- $\mu_0 = \delta_p^n$ where $p = f^n(p)$ is an **attracting periodic point**.

  In this case, f is called **regular**.

- $\mu_0 << m$ i.e. an **acip** (absolutely continuous invariant probability measure).

  In this case, f is called **stochastic**.

<u>Lyubich '02</u>

Almost every quadratic map is either regular or stochastic.

6. Criteria for limit thms of quadratic maps

- **<u>Bruin-Shen-van Strien '03</u>**

$$c \in \mathrm{Crit}(f), \quad |(f^n)'(f(c))| \to \infty \ (n \to \infty)$$

$$\Rightarrow \ \exists 1 \text{ acip } \mu_0$$

- **<u>Keller-Nowicki '92</u>, <u>Young '92</u>**

(CE) $c \in \mathrm{Crit}(f), \quad \liminf_{n \to \infty} \frac{1}{n} \log |(f^n)'(f(c))| > 0$

$\Rightarrow$ exponential decay of correlations

$\Rightarrow$ CLT i.e. $\sqrt{n}\left(\frac{1}{n} S_n \varphi - \int \varphi \, d\mu_0\right) \xrightarrow{d} N(0, \sigma^2) \ (n \to \infty)$ for $\varphi \in \mathrm{BV}$,

where $\sigma^2 := \int (\varphi_0)^2 \, d\mu_0 + 2 \sum_{n=1}^{\infty} \int \varphi_0 \cdot (\varphi_0 \circ f^n) \, d\mu_0 \in [0, +\infty)$,

$$\varphi_0 := \varphi - \int \varphi \, d\mu_0.$$

7. Local large deviations theorem

- **<u>Keller-Nowicki '92</u>**

(CE) ⇨ For $\varphi \in \mathrm{BV}$ and $0 < \varepsilon << 1$,

$$\exists \alpha_{\varphi}(\varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log m\left(\left| \frac{1}{n} S_n \varphi - \int \varphi \, d\mu_0 \right| \geq \varepsilon \right) < 0.$$

.

8. Previous result

- **<u>C - Takahasi '12, '14</u>**

(CE) + the slow recurrence condition

$$\lim_{n\to\infty} \frac{1}{n} \log |f^n(c) - c| = 0$$

⇨ $\forall \varphi \in C(X), \exists I_\varphi : \mathbb{R} \to [0, +\infty]$: lower semi-conti.

$$-\inf_{a \in \mathrm{int} A} I_\varphi(a) \leq \liminf_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n \varphi \in A\right)$$

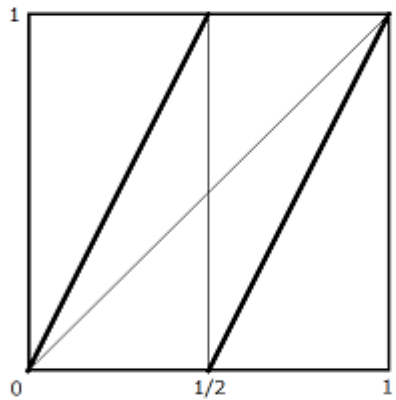
$$\leq \limsup_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n \varphi \in A\right) \leq -\inf_{a \in \mathrm{cl} A} I_\varphi(a)$$

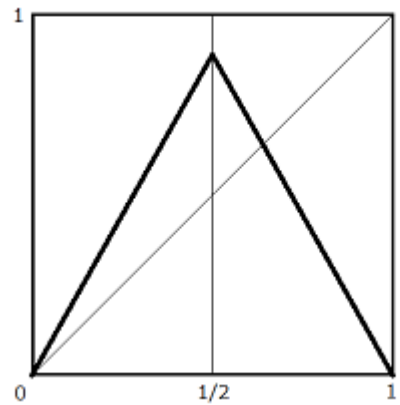
for any Borel set $A \subset \mathbb{R}$.

Indeed, we have obtained **LDP of level 2**.

9. Uniformly hyperbolic dynamical systems

$X := [0,1]$, m: Lebesgue measure





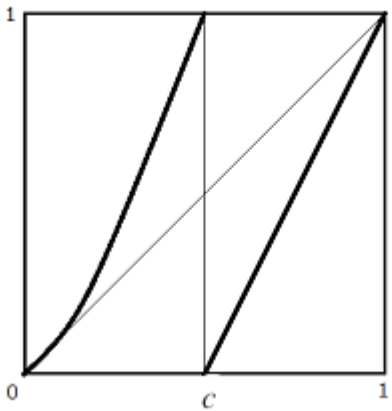
The Bernoulli map

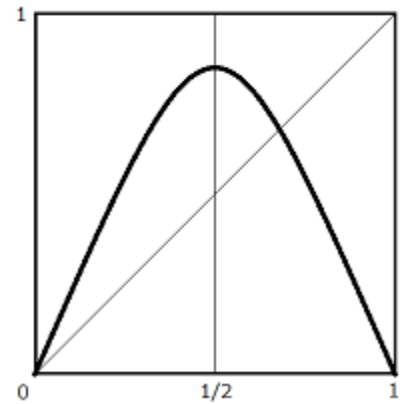
$$f(x) = kx \pmod 1$$
$$(k = 2, 3, \ldots)$$

The tent map

$$f(x) = \min\{ax, a(1-x)\}$$
$$(1 < a \leq 2)$$

10. Nonuniformly hyperbolic dynamical systems





The Manneville-Pomeau map

$$f(x) = x + x^{1+s} \pmod 1$$

$$(s > 0)$$

The quadratic map

$$f(x) = ax(1-x)$$

$$(1 < a \leq 4)$$

11. Induced map & LDP

We have obtained a **criterion to hold LDP** for non-uniformly hyperbolic dynamical systems which admit induced Markov maps. It is based on a **slope estimate** of the towers given by induced maps, and it is different from the tail estimate of Lai-Sang Young.

12. Tail & slope estimates (rough sketch)

- Tail estimate (Young '98) ⇨ acip, correlations, CLT etc.

$$a_k := \sum_{n=k}^{\infty} m(R \geq n) \to 0 \text{ how fast?}$$

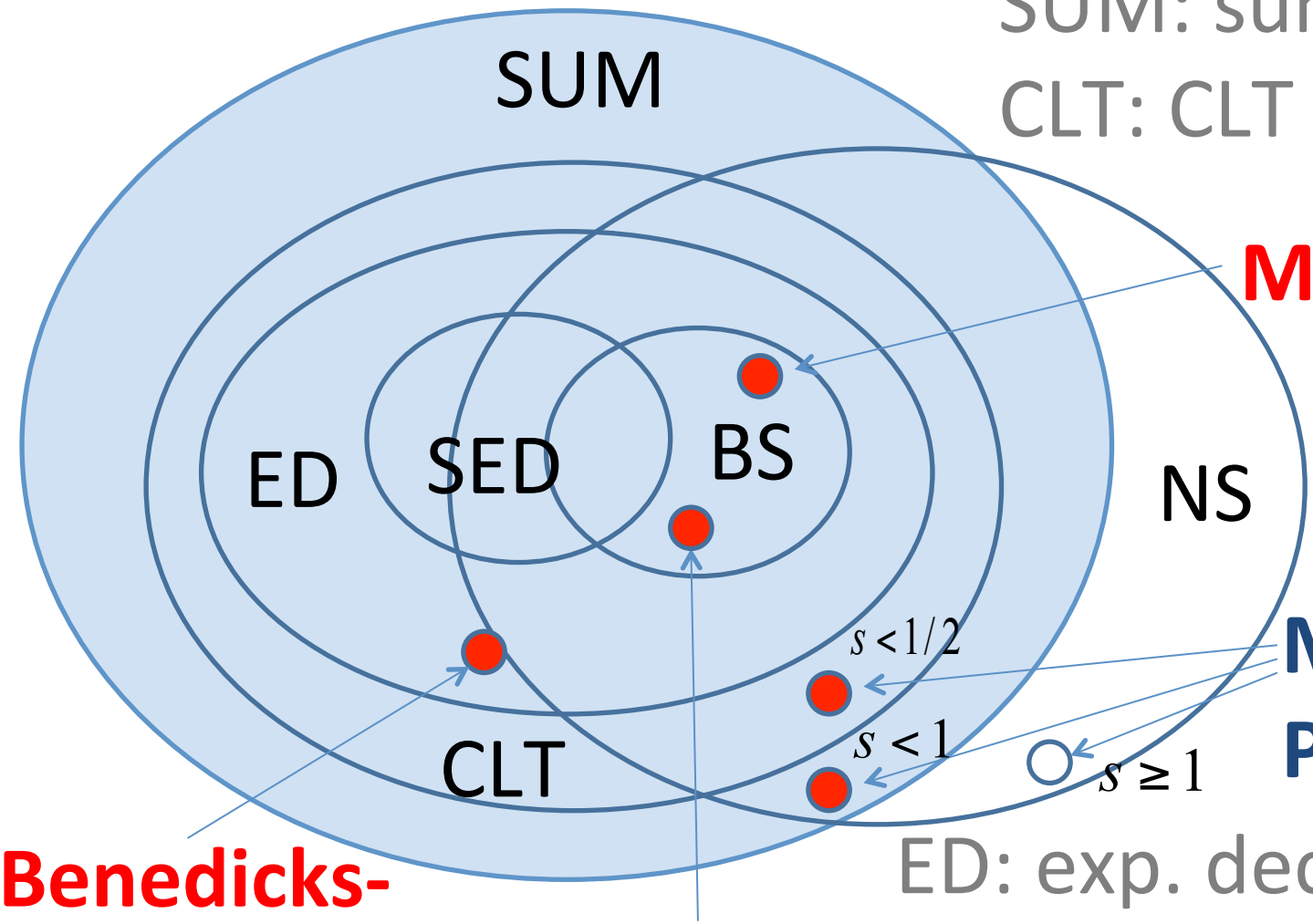
- Slope estimate (C'11) ⇨ **LDP**, MFA

$$b_k := m(R < k + l_k | R \geq k) \to 0$$

$$\text{how slow for some } l_k = o(k)?$$

13. ACIP exists

NS: nonsteep
BS: bounded slope
SUM: summable
CLT: CLT holds



Misiurewicz

Manneville-Pomeau

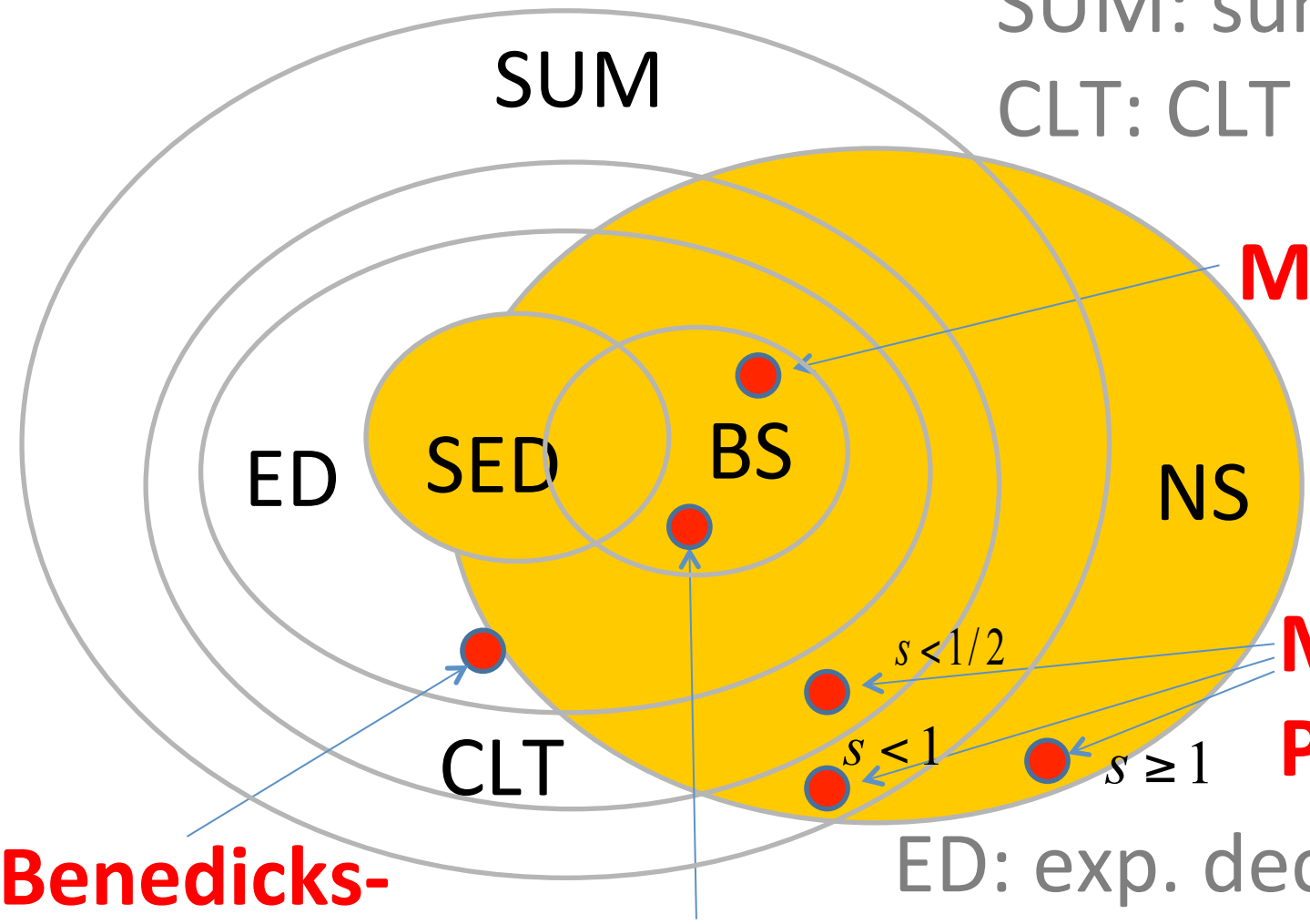
Benedicks-Carleson

unif.hyp.

ED: exp. decay
SED: super-exp. decay

14. LDP holds

NS: nonsteep
BS: bounded slope
SUM: summable
CLT: CLT holds



Misiurewicz

Manneville-Pomeau

Benedicks-Carleson

unif.hyp.

ED: exp. decay
SED: super exp. decay

15. Question

Is LDP universal in one-dimensional smooth dynamical systems?

More explicitly, does **any stochastic quadratic map satisfy LDP?** Or not?

16. Answer

Yes. The class of quadratic maps satisfying LDP is larger than that of stochastic ones.

And our result is also applicable to a class of multimodal maps with non-flat critical points.

17. Classification of quadratic maps

<u>Jonker-Rand '81</u>

Any S-unimodal map is one of the following 3 types:

1) an attracting periodic orbit exists;
2) Infinitely renormalizable;
3) **At most finitely renormalizable**.

Remark.

Any **stochastic** quadratic map is **at most finitely renormalizable**, and then **topologically exact** under suitable renormalization.

18. Topologically exactness

A continuous map $f : X \to X$ is **topologically exact** if

$$\phi \neq \forall J \subset X : \text{an interval}, \ \exists n \geq 1 \ \text{ s.t. } f^n(J) = X.$$

Remark.

- top. exact ⇨ specification ⇨ top. mixing.
  (no attracting periodic orbit, $\mathrm{cl}\,\mathrm{Per}(f) = X$,
  ergodic measures are entropy-dense in $\mathcal{M}_f$.)
- $f : C^3$ with $Sf < 0$ and top. exact
  ⇨ all periodic orbits are hyperbolic repelling.

19. Critical point and non-flatness

- A point $c \in X$ is a **critical point** of a differentiable map $f : X \to X$ if $f'(c) = 0$.

- A critical point $c \in X$ of f is **non-flat** if

  $\exists l > 1, \exists \phi, \psi : \mathbb{R} \to \mathbb{R}$: diffeos s.t.

$$\phi(c) = \psi \circ f(c) = 0 \text{ and } |\psi \circ f(x)| = |\phi(x)|^l$$

  for all x in a small neighborhood of c.

A continuously differentiable map has at most a finite number of non-flat critical points.

20. Definition of LDP

We say that $f : X \to X$ satisfies the **Large deviation principle (LDP)** (of level 2 for Lebesgue measure) if there exists a lower semi-continuous function $I : \mathcal{M} \to [0, +\infty]$ satisfying the following properties:

$$\liminf_{n\to\infty} \frac{1}{n} \log m(\delta_x^n \in \mathcal{G}) \geq -\inf_{\nu\in\mathcal{G}} I(\nu), \forall \mathcal{G} \subset \mathcal{M} : \text{open};$$

$$\limsup_{n\to\infty} \frac{1}{n} \log m(\delta_x^n \in \mathcal{C}) \leq -\inf_{\nu\in\mathcal{C}} I(\nu), \forall \mathcal{C} \subset \mathcal{M} : \text{closed},$$

where $\mathcal{M}$ denotes the space of Borel probability measures on X. The function I above is called the **rate function** if it exists. The rate fuction must vanish at a physical measure.

21. Main result

Theorem (C – Rivera·Letelier – Takahasi).

Let $f : X \to X$ be a **topologically exact** C^3 map having only hyperbolic repelling periodic orbits and non-flat critical points. Then f satisfies **LDP**, and the rate function $I : \mathcal{M} \to [0, +\infty]$ is given by

$$I(\mu) = -\inf_{\mathcal{G}} \sup\{F(\nu) : \nu \in \mathcal{G}\},$$ where

$$F(\nu) = \begin{cases} h(\nu) - \int \log |f'| d\nu & \text{if } \nu \in \mathcal{M}_f; \\ -\infty & \text{otherwise,} \end{cases}$$

$h(\nu)$ denotes the metric entropy, and the infimum is taken over all the neighborhoods $\mathcal{G}$ of $\mu \in \mathcal{M}$.

22. Remarks

- **No assumption on hyperbolicity for critical orbits** is needed in the theorem.

- The function F is not upper semi-continuous, so in general I is different from $-F$ (an example is given after the corollary).

23. Corollary (S-unimodal maps)

Any **at most finitely renormalizable S-unimodal map** satisfies **LDP** under suitable renormalization.

<u>The class of maps for which the corollary is applicable:</u>

① **stochastic** i.e. an acip exists;

② no acip, but a σ-finite acim exists (Johnson '87);

③ a wild Cantor attractor exists (Bruin-Keller-Nowicki-van Strien '96);

④ a physical measure is supported on a hyperbolic repelling fixed point (Hofbauer-Keller '90);

⑤ no physical measure & **LLN does not hold!** (Hofbauer-Keller '90)

24. An example that $I \neq -F$

<u>In the case ④</u>.

Hofbauer-Keller '90 have constructed a quadratic map for which the Dirac measure δ_p supported at a repelling fixed point p is physical. Then

$$I(\delta_p) = 0 \quad \text{but} \quad -F(\delta_p) = \log|f'(p)| > 0.$$

25. An example that LLN fails

For the example ⑤ that the physical measure does not exist (LLN fails), **the rate function seems to vanish at more than one** (and hence uncountable many) **invariant probability measures** supported on the closure of the critical orbit. And **almost every empirical distribution does not converge**, but oscillates between those measures.

On the other hand, the rate function does not vanish at any invariant probability measure whose support is isolated from the critical orbit.

"Averaged statistics hold, even for some systems without average asymptotics."

26. Idea of the proof

We construct a family of hyperbolic horseshoes (symbolic dynamics) by using distortion estimates with topologically exactness to show the theorem.

- <u>Lower bound</u>

  Pesin theory

  (a version of Katok horseshoe theorem for non-invertible maps)

- <u>Upper bound (hard)</u>

  Variational principle + **Uniform scale lemma**

27. Uniform scale lemma (key estimate)

Under the assumption of the theorem,

$\forall \varepsilon > 0, \exists \eta, \kappa, C > 0, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$,

$\forall V \subset X$: an interval with $\eta \leq |f^n(V)| \leq 2\eta$

$\exists W \subset V$: an interval, $\exists l \in \mathbb{N}$ s.t.

$|W| \geq e^{-\varepsilon n}|V|, \;\; n \leq l \leq n + C \log n,$

$f^l|_W : W \to f^l(W)$ is diffeomorphic

with distortion $\leq e^{\varepsilon n}, |f^l(W)| \geq \kappa.$

Thank you for your attention.